lines which cut off these values on the axes, or otherwise by the quadrilateral construction, if C be far away, provided the elevation is such a projection as would give the desired scenic effect for the point of view fixed on the x-axis. Unfortunately for so simple a construction, elevations are usually such projections as prohibit the selection of the point of view to best advantage, whereas in the present construction, the position of the point of sight is independent of the form of the orthogonal projection in which the elevation is given, as only the *distances* of its points from the horizontal plane are required. This permits the selection of the position of C and the direction of the central ray with reference to the plan alone, and practically gives power to select any point of view we choose to take. The drawing in the perspective is as condensed as could be desired, and is practically confined to the same space for all distances of C.

SOLUTION OF EXERCISES.

165

In any triangle ABC let a circle be inscribed touching the sides AB, BC, CA in N, L, M respectively. Let the centre O of this circle be joined to the vertices and from O let OP, OQ be drawn perpendicular respectively to OC, OB, and cutting BC in P and Q. Then if NP and AQ be drawn, these lines will be parallel, as will also AP and MQ. [*F. H. Loud.*]

SOLUTION.

$$\angle\, BOC = 90^\circ + \tfrac{1}{2}A, \text{ and } \angle\, C\,OP = 90^\circ;$$

$$\therefore\ \angle\, BOP = \tfrac{1}{2}A.$$

Also $$\angle\, OPB = 90^\circ + \tfrac{1}{2}C = \angle\, AOB.$$

Hence the triangles AOB and OPB are similar, and we have

$$\overline{BO}^2 = AB\,.\,BP. \tag{1}$$

Since $\frac{1}{2}B$ is a common angle in NBO and OBQ, these triangles are similar, and we have

$$\overline{BO}^2 = NB\,.\,BQ. \tag{2}$$

From (1) and (2) $\dfrac{AB}{NB} = \dfrac{BQ}{BP}$; $\therefore$ NP is parallel to AQ, which was to be proved.

[*W. O. Whitescarver.*]

168

If the normals at four points on a rectangular hyperbola meet in a point and the sum of the squares on the six distances between the four points, taken two together, is constant ($= k^2$), prove that the locus of the point of concourse of the normals is a circle. [*R. H. Graves.*]

SOLUTION I.

Let $x^2 - y^2 = a^2$ be the equation to the rectangular hyperbola, and (x, y) be the point of concourse of the normals. The co-ordinates of the four points are the roots of the biquadratics in x' and y'.

$$4x'^4 - 4xx'^3 + x'^2(x^2 - y^2 - 4a^2) + 4a^2xx' - a^2x^2 = 0, \quad (1)$$

$$4y'^4 - 4yy'^3 - y'^2(x^2 - y^2 - 4a^2) - 4a^2yy' + a^2y^2 = 0. \quad (2)$$

The sum of the squares of the differences of the roots of (1) is

$$3x^2 - 2(x^2 - y^2 - 4a^2).$$

The same function of the roots of (2) is

$$3y^2 + 2(x^2 - y^2 - 4a^2).$$

Hence the equation to the locus of the point of concourse is

$$3(x^2 + y^2) = k^2. \quad \text{[R. H. Graves.]}$$

SOLUTION II.

Let $XY = c^2$ be the equation to the curve; $(X - x)X + (Y + y)X = 0$ is the equation to the normal at the point XY.

Eliminating Y and X alternately by means of the given relation, we have

$$X^4 - xX^3 - c^2yX + c^4 = 0,$$

and

$$Y^4 - yY^3 - c^2xY + c^4 = 0.$$

Also,

$$\begin{aligned}&(x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_2 - x_3)^2 + (y_2 - y_3)^2 \\ &+ (x_3 - x_4)^2 + (y_3 - y_4)^2 + (x_4 - x_1)^2 + (y_4 - y_1)^2 \\ &\equiv 3\Sigma x_1^2 + 3\Sigma y_1^2 - 2\Sigma x_1x_2 - 2\Sigma y_1y_2 = k^2,\end{aligned}$$

and from the theory of equations,

$$\Sigma x_1 = x, \quad \Sigma y_1 = y, \quad \Sigma x_1x_2 = \Sigma y_1y_2 = 0.$$

x_1, x_2, x_3, x_4 being the four values of X, and y_1, y_2, y_3, y_4 the four values of Y, these conditions give

$$3\Sigma x_1^2 = x^2, \quad 3\Sigma y_1^2 = y^2.$$

Hence $3(x^2 + y^2) = k^2$, which is a circle; x and y being on each of the normals, and therefore on their point of concurse. [*E. Frisby.*]

169

FIND the locus of the instantaneous centre of a tangent to an ellipse when one point of the tangent moves in the auxiliary circle. [*R. H. Graves.*]

SOLUTION.

Find the locus of the intersection of the normal and the perpendicular on it from the focus; i. e. eliminate φ from

$$\frac{ax}{\cos\varphi} - \frac{by}{\sin\varphi} = a^2 - b^2,$$

and
$$\frac{b(x - ae)}{\sin\varphi} + \frac{ay}{\cos\varphi} = 0.$$

This gives $$(a^2 + b^2)\left(\frac{y}{x - ae} + \frac{x}{y}\right)^2 [y^2 + (x - ae)^2] = (a^2 - b^2)^2;$$

or
$$(a^2 + b^2)\left(\frac{y}{x} + \frac{x + ae}{y}\right)^2 (x^2 + y^2) = (a^2 - b^2)^2,$$

if the origin is moved to the point $(ae, 0)$. [*R. H. Graves.*]

178

FIND the point in which the normal to $xy = m^2$ cuts the curve again.

SOLUTION.

The normal at (x', y') has for its equation

$$xx' - yy' = x'^2 - y'^2. \tag{1}$$

By combining (1) and $xy = m^2$, the co-ordinates of the required point are found to be $-\frac{y'^2}{x'}, -\frac{x'^2}{y'}$.

Remark. The normal to $xy = m^2$ at the last point cuts the curve again at $\left(\frac{x'^5}{y'^4}, \frac{y'^5}{x'^4}\right)$.

The equation to the line joining this point and (x', y') is

$$xy'^5 + yx'^5 = x'y'^5 + y'x'^5 = m^2(x'^4 + y'^4).$$

The triangle enclosed by the three lines has an area equal to

$$\frac{(x'^2 + y'^2)(x'^4 - y'^4)(x'^6 + y'^6)}{2x'^5 y'^5}.$$ [*R. H. Graves.*]

179

FIND the area of the triangle formed by the asymptotes to an equilateral hyperbola and the normal to the curve.

SOLUTION.

The equation of the normal referred to the centre and asymptotes is

$$y - y' = \frac{x'}{y'}(x - x').$$

The half product of the values of x when $y = 0$, and y when $x = 0$, gives for the area of the triangle $(x'^2 - y'^2)^2/(2x'y')$. [*H. W. Draughon.*]

189

SHOW how to resolve a given force into three coplanar components acting in given lines not concurrent.

SOLUTION.

Let F be the force, *a point vector.* Since the lines are coplanar and non-current, they form a triangle. Let its vertices be e_1, e_2, e_3, and hence the three lines, e_1e_2, e_2e_3, e_3e_1. Then, if x_1, x_2, x_3 are scalar constants, we have

$$F = x_1e_2e_3 + x_2e_3e_1 + x_3e_1e_2.$$

Multiplying by e_1, $$e_1F = x_1e_1e_2e_3, \quad \text{or} \quad x_1 = \frac{e_1F}{e_1e_2e_3};$$

and similarly $$x_2 = \frac{e_2F}{e_1e_2e_3}, \quad x_3 = \frac{e_3F}{e_1e_2e_3}.$$

Hence $$F = \frac{1}{e_1e_2e_3}(e_1F \,.\, e_2e_3 + e_2F \,.\, e_3e_1 + e_3F \,.\, e_1e_2).$$

Thus the length of the component on e_1e_2 is the length of e_1e_2 times the ratio of the triangle formed by joining e_3 with the ends of F to the triangle $e_1e_2e_3$; and similarly for other components. [*E. W. Hyde.*]

EXERCISES.

213

FIND the centre of gravity of the area of one quadrant of the tetracuspid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. [*R. H. Graves.*]

214

AN equilateral hyperbola is circumscribed to a triangle. Find the greatest and least values of the transverse axis. [*R. H. Graves.*]

215

Two equal circles, radii r, intersect; find the average area common to both. [*Artemas Martin.*]